

STATISTICAL TOPOLOGY USING PERSISTENCE LANDSCAPES

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ABSTRACT. We define a new descriptor for persistent homology, which we call the persistence landscape, for the purpose of facilitating statistical inference. This descriptor may be thought of as an embedding of the usual descriptors, barcodes and persistence diagrams, into a space of functions, which inherits an L^p norm. We show that the corresponding metric is topologically equivalent to the $(p + 1)$ -Wasserstein distance, and that this metric space is complete and separable. We prove a stability theorem for persistence landscapes. For $p = 2$, we show that the Fréchet mean of persistence landscapes is the pointwise mean, and that the Fréchet variance is the integral of the pointwise variances. Furthermore, the sample mean of persistence landscapes converges pointwise to the mean of the underlying distribution, and there is a corresponding central limit theorem.

1. INTRODUCTION

1.1. Persistent homology. One of the main tools in applied topology is persistent homology. Let us start with a brief overview of one of the ways in which it is calculated. Data from an application is processed so that it may be considered to be a finite set of points in Euclidean space, or more generally, some Riemannian manifold. To this point cloud we apply a construction, such as the Čech complex, Vietoris-Rips complex, or alpha complex that results in a simplicial complex that depended on one parameter. For example, the Čech complex is the nerve of the union of balls of fixed radius centered at each of the data points.

As this parameter increases, our simplicial complex grows. We may consider the resulting object, X , as a finite sequence of simplicial complexes, together with the corresponding inclusion maps, or as a filtered simplicial complex. Taking simplicial homology with coefficients in some fixed field, we obtain a finite sequence of vector spaces, together with linear maps induced by the earlier inclusions.

$$(1) \quad M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n$$

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Homology classes which are in the images of these maps are referred to as persistent homology classes. Remarkably, there is an efficiently-computable complete invariant for (1), which is referred to as the *persistent homology* of X .

Persistent homology has two equivalent¹ standard descriptors: the *barcode*, consisting of a multiset of intervals, and the *persistence diagram*, consisting of a multiset of ordered pairs. Informally, an interval starting at a and ending at b , or an ordered pair (a, b) corresponds to a persistent homology class that is born at M_a and dies at M_b . For more details see the articles [6, 10], or the books [11, 5].

1.2. Metrics for persistent homology. Given two descriptors for persistent homology, one is immediately led to ask the following question. How close together are they? Because longer intervals correspond to homology classes that persist through a wider range of parameters, we would like our metric to give greater weight to longer intervals than to shorter ones. One suitable and commonly used metric is the *Wasserstein distance*, W_p , that has an “ L^p version” for each of $1 \leq p \leq \infty$ (Definition 3.1).

1.3. Probability and statistics. A good framework for probability theory is that of a *Polish space*. A metric space is a Polish space if and only if it is complete and separable. It has been shown [7] that the space of persistence diagrams with the Wasserstein distance is indeed a Polish space.

In the setting of a metric space, a good notion of mean and variance is given by the *Fréchet mean* and the *Fréchet variance* (Definition 5.8). Some nice existence results for Fréchet means of persistence diagrams were proven in [7]. One of the main motivations for the work in the present paper was to provide a setting in which one can calculate the Fréchet mean and variance for persistent homology. For a recent alternative approach, which gives an algorithm for calculating the Fréchet mean of a variant of the Wasserstein distance, see [9].

1.4. Persistence landscapes. In this paper, we describe a new descriptor for persistent homology which we call the *persistent landscape*. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$, the value of the persistence landscape of a barcode is the maximum radius (half the length) of an interval centered at t that is contained in k of the intervals in the barcode. It has a similar definition for a diagram of vector spaces and linear maps, such as (1).

¹In some other scenarios, the barcode can contain more information, as it allows one to differentiate between intervals that do or do not contain their endpoints.

The main technical advantage of this descriptor is that it is a real-valued function and thus the space of persistence landscapes inherits an L^p norm and corresponding metric. In addition, some of what we would like to calculate can be done in a pointwise fashion.

Furthermore, for each of the L^p norms, the persistence landscape gives, by definition, greater weight to longer intervals than to shorter ones. So it satisfies our main requirement for a metric for persistent homology.

1.5. Outline and summary of main results. We start by defining persistence landscapes and giving some basic results in Section 2.

In the third section we prove that for finite persistence diagrams, the metric given by the L^p distance between the corresponding persistence landscapes is topologically equivalent to the $(p + 1)$ -Wasserstein distance (Theorem 3.2).

In Section 4, we show that the persistence landscape of a tame Lipschitz function on a suitably nice space is stable with respect to the supremum norm (Theorem 4.1).

In the fifth section we prove (Theorem 5.3) that the space of persistence landscapes with the metric inherited from the L^p norm is a Polish space. We also prove (Theorem 5.15) that for $p = 2$, the Fréchet mean of persistence landscapes is just the pointwise mean, and that the Fréchet variance is the integral of the pointwise variances. From the strong law of large numbers, we get that the sample mean of persistence landscapes converges pointwise to the Fréchet mean of the underlying distribution (Corollary 5.17). From the central limit theorem, we get a pointwise central limit theorem for the sample mean of persistence landscapes (Corollary 5.19).

We end by applying these results to points sampled from a torus and a sphere in Section 6.

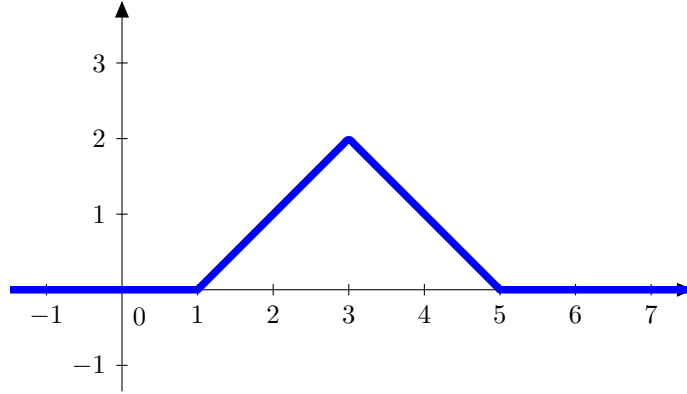
1.6. Notation. Let us fix some notation. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We denote the domain and codomain of a function f , by $\text{dom}(f)$ and $\text{cod}(f)$, respectively.

2. PERSISTENCE LANDSCAPES

2.1. Persistence landscapes from persistence diagrams and barcodes. In this section we describe the persistence landscape corresponding to a finite persistence diagram and a finite barcode.

A *persistence diagram*² [2] is a multiset of pairs of real numbers (a, b) with $a \leq b$. A *barcode* [10] is a multiset of intervals. Here we consider a

²In [2] this is called a reduced persistence diagram.

FIGURE 1. The graph of $f_{(1,5)}$.

finite persistence diagram, $\{(a_i, b_i)\}_{i=1}^n$, with $a_i \leq b_i$, or a finite barcode, $\{I_i\}_{i=1}^n$. In the second case we use the endpoints $a_i \leq b_i$ of interval I_i .

First we give an informal description of the persistence landscape of a persistence diagram. For each $i \in \{1, \dots, n\}$, draw an isosceles triangle with base the interval $[a_i, b_i]$ on the horizontal t -axis, and sides with slope 1 and -1 . So the triangle has vertices $(a_i, 0)$, $(b_i, 0)$ and $(\frac{a_i+b_i}{2}, \frac{b_i-a_i}{2})$. This subdivides the plane into a number of polygonal regions. Label each of these regions by the number of triangles containing it. For $k \in \mathbb{N}$, let P_k be the union of the polygonal regions with values at least k . Let $\lambda_k : \mathbb{R} \rightarrow \mathbb{R}$ be the function whose graph is the upper contour of P_k , with $\lambda_k(a) = 0$ if the vertical line $t = a$ does not intersect P_k .

Next we give a more formal description. Given (a, b) with $a \leq b$, define $f_{(a,b)} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(2) \quad f_{(a,b)}(t) = \min(t - a, b - t)_+,$$

where c_+ denotes $\max(c, 0)$. Note that if $m = \frac{a+b}{2}$ and $h = \frac{b-a}{2}$, then

$$f_{(a,b)}(t) = (h - |t - m|)_+.$$

That is, $f_{(a,b)}$ is the piecewise linear bump function with height h and bandwidth h , centered at m . See Figure 1 for an example.

Now define the *persistence landscape* of $\{(a_i, b_i)\}_{i=1}^n$ to be the set of functions $\lambda_k : \mathbb{R} \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, given by

$$\lambda_k(t) = k\text{th largest value of } \{f_{(a_i, b_i)}(t)\}_{i=1}^n,$$

with $\lambda_k(t) = 0$ if $k > n$. See Figure 2 for an example.

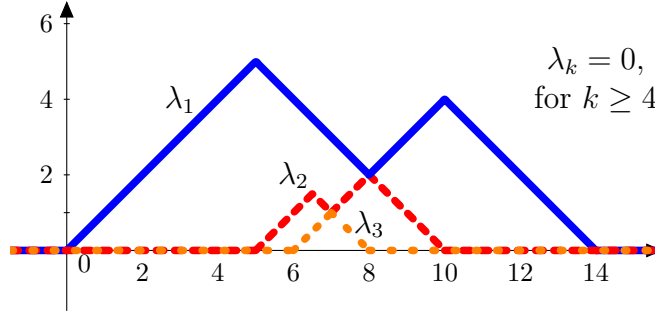


FIGURE 2. The graph of the persistence landscape of $\{(0, 10), (5, 8), (6, 14)\}$.

Finally we give some motivation for this construction. We may think of the process of replacing a barcode with a persistence landscape as replacing each interval in a barcode with the set of all of its subintervals. To make this more precise, let us change the coordinates from (a, b) to (m, h) , where $m = \frac{a+b}{2}$ and $h = \frac{b-a}{2}$. We are changing from birth – death coordinates to mid-lifetime (mid-life) – half time alive (half-life) coordinates. The union of all points (m', h') corresponding to intervals $I' \subseteq I$ is the isosceles triangle with vertices $(a, 0)$, (m, h) and $(b, 0)$. Then

$$\lambda_k(t) = \sup\{h \mid \text{the interval centered at } t \text{ with radius } h \text{ is contained in } k \text{ intervals in the barcode}\}.$$

2.2. Diagrams of vector spaces indexed by (\mathbb{R}, \leq) . Here we review a framework for persistent homology developed in [1].

Let M be a *diagram of finite dimensional vector spaces indexed by (\mathbb{R}, \leq)* . That is, for each $a \in \mathbb{R}$, $M(a)$ is a finite dimensional vector space over some field, \mathbb{F} , and for all $a \leq b$, we have a linear map $M(a) \rightarrow M(b)$, denoted $M(a \leq b)$, satisfying $M(a \leq a) = \text{Id}_{M(a)}$ for all $a \in \mathbb{R}$, and for all $a \leq b \leq c$, $M(a \leq c) = M(b \leq c) \circ M(a \leq b)$ ³. We write $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$.

For $p \geq 0$, define the p -persistence of $M(a)$ to the image of the map $M(a \leq a + p)$. For $a \leq b$, define the corresponding *Betti number* of M , by

$$\beta^{a,b}(M) = \dim(\text{im}(M(a \leq b))).$$

³That is, M is a functor from the poset (\mathbb{R}, \leq) to the category of finite-dimensional vector spaces.

For an interval $I \subseteq \mathbb{R}$, we define $\chi_I \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ by

$$(3) \quad \chi_I(c) = \begin{cases} \mathbb{F}, & \text{if } c \in I \\ 0, & \text{if } c \notin I, \end{cases} \quad \text{and} \quad \chi_I(c \leq d) = \begin{cases} \text{Id}_{\mathbb{F}}, & \text{if } c, d \in I, \\ 0, & \text{otherwise.} \end{cases}$$

For a multiset $S = \{I_i\}_{i=1}^n$, where I_i is an interval, define $\chi_S \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ by

$$(4) \quad \chi_S(c) = \oplus_{i=1}^n \chi_{I_i}(c), \quad \text{and} \quad \chi_S(c \leq d) = \oplus_{i=1}^n \chi_{I_i}(c \leq d).$$

Say that $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ has *finite type* if $M \cong \chi_S$ for some $S = \{I_i\}_{i=1}^n$. Write $\mathbf{Vec}_{\text{ft}}^{(\mathbb{R}, \leq)}$ for the set of all elements in $\mathbf{Vec}^{(\mathbb{R}, \leq)}$ that have finite type. In [1], it is shown that such an isomorphism is unique up to reordering. Thus,

Theorem 2.1 ([1, Corollary 4.9]). *There is a bijection between finite barcodes and isomorphism classes of finite type diagrams, given by $S = \{I_i\}_{i=1}^n \mapsto \chi_S$.*

If $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ has finite type, let $I(M)$ be the inverse image of this bijection. For an interval I , let $x(I) = (\inf(I), \sup(I)) \in \overline{\mathbb{R}}^2$. For a multiset of intervals, $S = \{I_i\}$, let $D(S)$ denote the multiset $\{x(I_i)\}$. Let $D(M)$ denote $D(I(M))$.

It is also shown in [1], that the finite-type diagrams are exactly those with finitely many *critical values*.

2.3. Persistence landscapes from diagrams. Let us start with the observation that for a fixed diagram M and fixed $t \in \mathbb{R}$, $\beta^{t-\bullet, t+\bullet}(M)$ is a decreasing function. That is,

Lemma 2.2. *Let $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ and $t \in \mathbb{R}$. Then for $0 \leq s \leq s'$,*

$$\beta^{t-s', t+s'}(M) \leq \beta^{t-s, t+s}(M).$$

Proof. Since $t - s' \leq t - s \leq t + s \leq t + s'$, $M(t - s' \leq t + s') = M(t + s \leq t + s') \circ M(t - s \leq t + s) \circ M(t - s' \leq t - s)$. It follows that $\beta^{t-s', t+s'}(M) \leq \beta^{t-s, t+s}(M)$. \square

Definition 2.3. Let $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$. The *persistence landscape* of M is the sequence of functions $\{\lambda_k(M) : \mathbb{R} \rightarrow \overline{\mathbb{R}}\}_{k \in \mathbb{N}}$ given by

$$\lambda_k(M)(t) = \sup(s \geq 0 \mid \beta^{t-s, t+s}(M) \geq k).$$

Define $\lambda(M) : \mathbb{N} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ by $\lambda(M)(k, t) = \lambda_k(M)(t)$.

In the next two examples, we show that in light of Theorem 2.1, this definition does generalize the definition in Section 2.1.

Example 2.4. Let I be an interval with endpoints $-\infty \leq a \leq b \leq \infty$. We will calculate the persistence landscape of the diagram χ_I , defined in (3).

Since $\dim(\chi_I(c)) \leq 1$ for all $c \in \mathbb{R}$, for each $k \geq 2$, $\lambda_k(\chi_I) = 0$. For the remaining calculation,

$$(5) \quad \lambda_1(\chi_I(t)) = \sup(s \geq 0 \mid \dim(\text{im}(\chi_I(t-s \leq t+s))) \geq 1).$$

If $t \notin I$, then $\chi_I(t) = 0$ and since $\chi_I(t-s \leq t+s) = \chi(t \leq t+s) \circ \chi(t-s \leq t)$, it follows that the right hand side of (5) equals 0. If $t \in I$, then

$$\begin{aligned} \lambda_1(\chi_I)(t) &= \sup(s \geq 0 \mid t-s, t+s \in I) \\ &= \min(t-a, b-t). \end{aligned}$$

Therefore, $\lambda_1(\chi_I)(t) = \min(t-a, b-t)_+ = f_{(a,b)}(t)$ (see (2)). \square

Example 2.5. Let $S = \{I_i\}_{i=1}^n$, where I_i is an interval with endpoints $-\infty \leq a_i \leq b_i \leq \infty$. Consider the diagram $\chi_S \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$, defined in (4). Then,

$$\begin{aligned} \lambda_k(\chi_S)(t) &= \sup(s \geq 0 \mid \beta^{t-s, t+s}(\chi_S) \geq k) \\ &= \sup\left(s \geq 0 \mid \sum_{i=1}^n \beta^{t-s, t+s}(\chi_{I_i}) \geq k\right). \end{aligned}$$

If $k > n$, then the right hand side is 0. Otherwise,

$$\begin{aligned} \lambda_k(\chi_S)(t) &= k\text{th largest element of } \sup(s \geq 0 \mid \beta^{t-s, t+s}(\chi_{I_i}) \geq 1) \\ &= k\text{th largest element of } \lambda_1(\chi_{I_i})(t) \\ &= k\text{th largest element of } f_{(a_i, b_i)}(t). \end{aligned}$$

\square

Next we establish some basic properties of the persistence landscape.

Lemma 2.6. $\lambda_k(M) : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ satisfies the following properties:

- (1) $\lambda_k(M)(t) \geq 0$,
- (2) $\lambda_k(M)(t) \geq \lambda_{k+1}(M)(t)$, and
- (3) $\lambda_k(M)(t)$ is 1-Lipschitz.

Proof. The first two properties in Lemma 2.6 follow immediately from Definition 2.3. It remains to prove that $\lambda_k(M)(t)$ is 1-Lipschitz. That is, $|\lambda_k(M)(t) - \lambda_k(M)(s)| \leq |t-s|$, for all $s, t \in \mathbb{R}$.

Let $s, t \in \mathbb{R}$. Without loss of generality, assume that $\lambda_k(M)(t) \geq \lambda_k(M)(s)$. If $\lambda_k(M)(t) \leq |t-s|$, then $\lambda_k(M)(t) - \lambda_k(M)(s) \leq \lambda_k(M)(t) \leq |t-s|$ and we are done. So assume that $\lambda_k(M)(t) > |t-s|$.

Let $0 < \varepsilon < \lambda_k(M)(t) - |t - s|$. By Definition 2.3 and Lemma 2.2,

$$(6) \quad \beta^{t - \lambda_k(M)(t) + \varepsilon, t + \lambda_k(M)(t) - \varepsilon} \geq k.$$

Since $-|t - s| \leq t - s \leq |t - s|$, $s - |t - s| \leq t \leq s + |t - s|$. Also, $-\lambda_k(M)(t) + |t - s| + \varepsilon < 0 < \lambda_k(M)(t) - |t - s| - \varepsilon$. Combining these, we have,

$$\begin{aligned} t - \lambda_k(M)(t) + \varepsilon &\leq s - \lambda_k(M)(t) + |t - s| + \varepsilon < s < \\ &< s + \lambda_k(M)(t) - |t - s| - \varepsilon \leq t + \lambda_k(M)(t) - \varepsilon. \end{aligned}$$

Together with (6) and Lemma 2.2, we see that

$$\beta^{s - \lambda_k(M)(t) + |t - s| + \varepsilon, s + \lambda_k(M)(t) - |t - s| - \varepsilon} \geq k.$$

Therefore $\lambda_k(M)(s) \geq \lambda_k(M)(t) - |t - s|$. Thus $\lambda_k(M)(t) - \lambda_k(M)(s) \leq |t - s|$. \square

To take advantage of the linear structure of the set of functions from $\mathbb{N} \times \mathbb{R}$ to $\overline{\mathbb{R}}$, we enlarge the set of persistence landscapes to its convex hull.

Definition 2.7. Let PL be the convex hull of the set $\{\lambda(M) \mid M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}\}$.

The following characterization of functions in PL follows from the definitions.

Lemma 2.8. *The functions $f : \mathbb{N} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ in PL satisfy the following properties:*

- (1) $f(k, t) \geq 0$,
- (2) $f(k, t) \geq f(k + 1, t)$, and
- (3) $f(k, t)$ is 1-Lipschitz for fixed k . \square

To help visualize the graph of $f : \mathbb{N} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$, we can extend it to a function $\bar{f} : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ by setting

$$(7) \quad \bar{f}(x, t) = \begin{cases} f(\lceil x \rceil, t), & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

See Figure 3 for an example.

2.4. Norms and metrics for persistence landscapes. Let (X, Σ, μ) be a measure space and let $f : X \rightarrow \overline{\mathbb{R}}$. Recall that for $1 \leq p < \infty$, $\|f\|_p = [\int |f|^p d\mu]^{\frac{1}{p}}$, and $\|f\|_\infty = \text{ess sup } f = \inf\{a \mid \mu\{s \in X \mid f(s) > a\} = 0\}$. For $1 \leq p \leq \infty$, $\mathcal{L}^p(X) = \{f : X \rightarrow \overline{\mathbb{R}} \mid \|f\|_p < \infty\}$. The

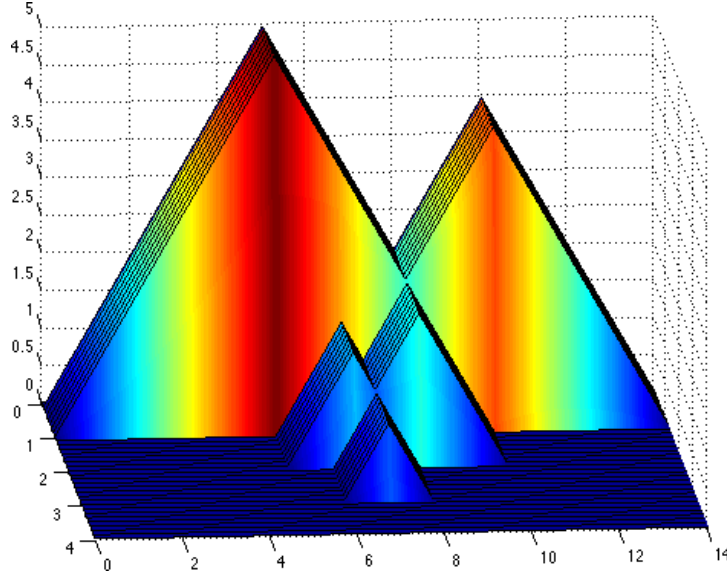


FIGURE 3. A 3D version of the graph of the persistence landscape in Figure 2.

seminorm $\|-\|_p$ on $\mathcal{L}^p(X)$ induces a pseudometric on $\mathcal{L}^p(X)$ given by $d(f, g) = \|f - g\|_p$.

On $\mathbb{N} \times \mathbb{R}$, we use the measure that is the product of the counting measure on \mathbb{N} and the Lebesgue measure on \mathbb{R} . For $f : \mathbb{N} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$,

$$\|f\|_p^p = \sum_{k=1}^{\infty} \|f_k\|_p^p,$$

where $f_k(t) = f(k, t)$. If we extend f to $\bar{f} : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$, as in (7), we have $\|f\|_p = \|\bar{f}\|_p$, where we use the Lebesgue measure on \mathbb{R}^2 for the latter.

The following corollary follows from Property (3) of Lemma 2.8, which implies that $f(k, t)$ is continuous for all k .

Corollary 2.9. *Let $f \in PL$. If $\|f\|_p = 0$ then $f = 0$.* \square

For $1 \leq p \leq \infty$, define

$$PL^p = PL \cap \mathcal{L}^p(\mathbb{N} \times \mathbb{R}).$$

By Corollary 2.9, the pseudometric on $\|-\|_p$ induced by the seminorm $\|-\|_p$ is a metric on PL^p . We will consider PL^p as a metric space with this metric, $d(f, g) = \|f - g\|_p$.

3. PERSISTENCE LANDSCAPES AND METRICS

In this section we relate the metric induced by the L^p norm to the Wasserstein distance.

For $1 \leq p \leq \infty$, the metric induced by $\|-\|_p$ on PL^p induces an (extended) pseudometric on $\mathbf{Vec}^{(\mathbb{R}, \leq)}$ via the function λ in Definition 2.3. Call this the *p-persistence landscape distance*, denote d_p . That is, for $M_1, M_2 \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$,

$$d_p(M_1, M_2) = \|\lambda(M_1) - \lambda(M_2)\|_p.$$

For $1 \leq p \leq \infty$, Cohen-Steiner, Edelsbrunner, Harer and Mileyko define a metric W_p , called the *p*th Wasserstein distance on the set of finite persistence diagrams [3]. For $p = \infty$, this is same as the bottleneck distance defined in [2]. Here, we use Theorem 2.1 to define it for finite type diagrams in $\mathbf{Vec}^{(\mathbb{R}, \leq)}$.

First, we need some notation. For $(a, b) \in \overline{\mathbb{R}}^2$, let $\|(a, b)\|_\infty = \max(|a|, |b|)$. For a finite type persistence diagram M , let $\overline{D}(M)$ denote the union of the multiset $D(M)$ (see Section 2.2) together with the multiset consisting of the points in $\overline{\Delta} := \{(a, a) \mid a \in \overline{\mathbb{R}}\}$ each with infinite multiplicity.

Definition 3.1. Assume M, M' are finite type diagrams in $\mathbf{Vec}^{(\mathbb{R}, \leq)}$. For $1 \leq p < \infty$, let

$$W_p(M, M')^p = \min_{f: \overline{D}(M) \xrightarrow{\cong} \overline{D}(M')} \sum_{x \in \text{dom}(f)} \|x - f(x)\|_\infty^p, \text{ and let}$$

$$W_\infty(M, M') = \min_{f: \overline{D}(M) \xrightarrow{\cong} \overline{D}(M')} \max_{x \in \text{dom}(f)} \|x - f(x)\|_\infty.$$

Similarly, we can define W_p and W_∞ for finite persistence diagrams.

The main result of this section is the following. It will follow directly from Corollaries 3.12 and 3.14.

Theorem 3.2. *The metrics d_p and W_{p+1} are topologically equivalent on finite persistence diagrams. Equivalently, the pseudometrics d_p and W_{p+1} are topologically equivalent on finite-type (\mathbb{R}, \leq) -indexed diagrams of finite-dimensional vector spaces.*

As a warm up for the proof of this theorem, we prove the following two easier equalities.

Theorem 3.3. *Let $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ have finite type. Then for $1 \leq p < \infty$,*

$$\|\lambda(M)\|_p^p = \frac{2}{p+1} W_{p+1}(M, \emptyset)^{p+1}.$$

The right hand side in the above theorem can also be written in terms of *total p-persistence*, defined in [3]. For $1 \leq p < \infty$, let

$$(8) \quad \text{pers}_p(M) = \sum_{I \in I(M)} (\sup(I) - \inf(I))^p.$$

We remark that

$$W_p(M, \emptyset)^p = \frac{1}{2^p} \text{pers}_p(M).$$

So the conclusion of Theorem 3.3 can also be written as,

$$\|\lambda(M)\|_p^p = \frac{1}{2^p(p+1)} \text{pers}_{p+1}(M).$$

Proof. Assume $M \cong \chi_S$, where $S = \{I_i\}_{i=1}^n$ and I_i is an interval. First,

$$\|\lambda(\chi_I)\|_p^p = \|f_I\|_p^p = 2 \int_0^{\frac{b-a}{2}} t^p dt = \frac{2}{p+1} \left(\frac{b-a}{2} \right)^{p+1} = \frac{2}{p+1} W_{p+1}(\chi_I, \emptyset).$$

Next, by appropriately decomposing the domain of integration,

$$\|\lambda(M)\|_p^p = \sum_{i=1}^n \|\lambda(\chi_{I_i})\|_p^p = \frac{2}{p+1} W_{p+1}(M, \emptyset)^{p+1}. \quad \square$$

Theorem 3.4. *Let $M \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ have finite type. Then*

$$\|\lambda(M)\|_\infty = W_\infty(M, \emptyset).$$

Proof. Assume $M \cong \chi_S$, where $S = \{I_i\}_{i=1}^n$. For $1 \leq i \leq n$, let $(a_i, b_i) = (\inf(I_i), \sup(I_i))$ and $h_i = \frac{b_i - a_i}{2}$. Then $\|f_{(a_i, b_i)}\|_\infty = h_i$. It follows that $\|\lambda(M)\|_\infty = \max_{1 \leq i \leq n} h_i = W_\infty(M, \emptyset)$. \square

We now establish a number of preliminary results that we will need in order to prove Theorem 3.2, which will finally follow directly from Corollaries 3.12 and 3.14.

Proposition 3.5. *Let $M, M' \in \mathbf{Vec}_{\text{ft}}^{(\mathbb{R}, \leq)}$. Then,*

$$\|\lambda(M) - \lambda(M')\|_\infty \leq W_\infty(M, M').$$

To prove this proposition, we will use the following two lemmas.

Lemma 3.6. (1) *Let I be a finite interval. Let $x = (\inf I, \sup I) \in \mathbb{R}^2$, and $x' = (\frac{\inf I + \sup I}{2}, \frac{\inf I + \sup I}{2})$. Then $\|\lambda(\chi_I)\|_\infty = \|x - x'\|_\infty$.*
 (2) *Let I and I' be finite intervals. Let $x = (\inf I, \sup I)$ and $x' = (\inf I', \sup I')$. Then $\|\lambda(\chi_I) - \lambda(\chi_{I'})\|_\infty \leq \|x - x'\|_\infty$.*

Proof. (1) Recall that $\lambda(\chi_I)(1, t) = f_x(t)$. So $\|\lambda(\chi_I)\|_\infty = \|f_x\|_\infty = \frac{\sup I - \inf I}{2} = \|x - x'\|_\infty$.

$$(2) \quad \|\lambda(M) - \lambda(M')\|_\infty = \sup_t |\sup(s \geq 0 \mid a \leq t - s \leq t + s \leq b) - \sup(s \geq 0 \mid a' \leq t - s \leq t + s \leq b')| \leq \max(|a - a'|, |b - b'|) = \|x - x'\|_\infty. \quad \square$$

Given a sequence c_1, \dots, c_n , let $c_{(1)}, \dots, c_{(n)}$ denote a reordering of the sequence so that $c_{(1)} \leq \dots \leq c_{(n)}$.

Lemma 3.7. *If $|c_i - d_i| \leq \delta$ for $i = 1, \dots, n$, then $|c_{(k)} - d_{(k)}| \leq \delta$ for $k = 1, \dots, n$.*

Proof. Let $k \in \{1, \dots, n\}$. Since $c_{(1)} \leq \dots \leq c_{(k)}$, using the assumption k times, we see that

$$\#\{1 \leq i \leq n \mid d_i \leq c_{(k)} + \delta\} \geq k.$$

Similarly,

$$\#\{1 \leq i \leq n \mid c_i \leq d_{(k)} + \delta\} \geq k.$$

It follows that $d_{(k)} \leq c_{(k)} + \delta$ and $c_{(k)} \leq d_{(k)} + \delta$. Therefore $|c_{(k)} - d_{(k)}| \leq \delta$. \square

Proof of Proposition 3.5. Assume $M \cong \chi_S$, where S is the multiset $\{I_i\}_{i=1}^n$. Let $\delta = W_\infty(M, M')$. Let $t \in \mathbb{R}$. Then $\{\lambda(M)(k, t)\}_{k=1}^\infty = \{\lambda(\chi_{I_i})(1, t)\}_{i=1}^n \cup \{0\}$ and similarly for $\lambda(M')(k, t)$. By the definition of W_∞ and Lemma 3.6, there exists a bijection between $\{\lambda(M)(k, t)\}$ and $\{\lambda(M')(k, t)\}$ such that the absolute value of the difference between corresponding elements is less than or equal to δ . By Lemma 3.7, $|\lambda(M)(k, t) - \lambda(M')(k, t)| \leq \delta$ for all k . \square

Lemma 3.8. *Let $M, M' \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$ and $1 \leq p < \infty$. Then*

$$W_\infty(M, M') \leq W_p(M, M').$$

Proof. Fix $f : \overline{D}(M) \cong \overline{D}(M')$. Let $A = \left[\sum_{x \in \text{dom}(f)} \|x - f(x)\|_\infty^p \right]^{\frac{1}{p}}$.

Let $x \in \text{dom}(f)$. Then $A \geq [\|x - f(x)\|_\infty^p]^{\frac{1}{p}} = \|x - f(x)\|_\infty$. Therefore $\sup_{x \in \text{dom}(f)} \|x - f(x)\|_\infty \leq A$. The result follows. \square

Lemma 3.9. *Let $M \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$. Then,*

$$\mu(\{(k, t) \mid \lambda(M)(k, t) > 0\}) = 2W_1(M, \emptyset).$$

Proof. $2W_1(M, \emptyset) = 2 \sum_{I \in I(M)} \frac{\sup I - \inf I}{2} = \sum_{I \in I(M)} (\sup I - \inf I)$. The result follows from rearranging the domain of $\lambda(M)$. \square

Proposition 3.10. *Assume that $M, M' \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ have finite type. Then*

$$\|\lambda(M) - \lambda(M')\|_p \leq 2W_1(M, \emptyset)W_\infty(M, M')^p + \frac{2}{p+1}W_{p+1}(M, M')^{p+1}.$$

Proof. Let $f : \overline{D}(M) \xrightarrow{\cong} \overline{D}(M')$ be the bijection attaining the minimum in the definition of $W_{p+1}(M, M')$. Now,

$$\begin{aligned} \|\lambda(M) - \lambda(M')\|_p^p &= \int (\lambda(M) - \lambda(M'))^p d\mu \\ &= \int_{\lambda(M)(k,t) > 0} (\lambda(M) - \lambda(M'))^p d\mu + \int_{\substack{\lambda(M')(k,t) > 0 \\ \lambda(M)(k,t) = 0}} (\lambda(M'))^p d\mu \end{aligned}$$

Combining Proposition 3.5 and Lemma 3.9, we see that the left hand term is bounded by $2W_1(M, \emptyset)W_\infty(M, M')^p$.

The right hand term is bounded by $\sum_{x \in \text{cod}(f) - \overline{\Delta}} 2 \int_0^{\|x - f^{-1}(x)\|_\infty} t^p dt \leq \sum_{x \in \text{dom}(f)} \frac{2}{p+1} \|x - f(x)\|_\infty^{p+1} = \frac{2}{p+1} W_{p+1}(M, M')^{p+1}$. \square

Corollary 3.11. *Assume that $M, M' \in \mathbf{Vec}^{(\mathbb{R}, \leq)}$ have finite type and that $W_{p+1}(M, M') < W_1(M, \emptyset)$. Then*

$$(9) \quad \|\lambda(M) - \lambda(M')\|_p \leq \left\{ 2W_1(M, \emptyset) \left[1 + \frac{1}{p+1} \right] \right\}^{\frac{1}{p}} W_{p+1}(M, M').$$

Proof. By Lemma 3.8, the left hand term on the right side of the inequality in Proposition 3.10 is bounded by $2W_1(M, \emptyset)W_{p+1}(M, M')^p$. We thus have,

$$\begin{aligned} \|\lambda(M) - \lambda(M')\|_p^p &\leq 2W_1(M, \emptyset)W_{p+1}(M, M')^p + \frac{2}{p+1} W_{p+1}(M, M')^{p+1} \\ &\leq 2W_1(M, \emptyset) \left[1 + \frac{1}{p+1} \right] W_{p+1}(M, M')^p. \quad \square \end{aligned}$$

Corollary 3.12. *Let $M \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$. Let $r > 0$. Then there exists r' such that $B_{r'}(M; W_{p+1}) \subseteq B_r(M; d_p)$.*

Proof. Let $k_{M,p}$ denote the coefficient of $W_{p+1}(M, M')$ in (9). Let $r' = \min(W_1(M, \emptyset), \frac{r}{k_{M,p}})$. Choose $M' \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$ such that $W_{p+1}(M, M') < r'$. Then by Corollary 3.11,

$$\|\lambda(M) - \lambda(M')\|_p \leq k_{M,p} W_{p+1}(M, M') < k_{M,p} r' \leq r. \quad \square$$

Proposition 3.13. *Let $M, M' \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$. If $\|\lambda(M) - \lambda(M')\|_p$ is sufficiently small, then*

$$\|\lambda(M) - \lambda(M')\|_p^p \geq \frac{p+3}{(p+1)2^{p+1}} W_{p+1}(M, M')^{p+1}.$$

Proof. Consider $a \leq b$, $a' \leq b'$ with $\varepsilon = \|(a, b) - (a', b')\|_\infty$. Assume that $\varepsilon \ll b - a$. Recall that $(m, h) = (\frac{a+b}{2}, \frac{b-a}{2})$ and similarly for (m', h') . So $\|(m, h) - (m', h')\|_1 = \varepsilon$.

We are interested in $f_{(a,b)}$ and $f_{(a',b')}$. Let \mathcal{B} , be the radius ε box in the 1-norm centered at (m, h) . Then (m', h') lies on the boundary of this box. Consider the intersection of \mathcal{B} and the symmetric difference of the regions under the graphs $f_{(a,b)}$ and $f_{(a',b')}$. This region corresponds to part of the area between the curves $f_{(a,b)}$ and $f_{(a',b')}$. A minimizer for the contribution of this region to the integral $\|f_{(a,b)} - f_{(a',b')}\|_p^p$ is the point $(m', h') = (m + \frac{3}{4}\varepsilon, h + \frac{1}{4}\varepsilon)$. From this we calculate, $\|f_{(a,b)} - f_{(a',b')}\|_p^p \geq \frac{p+3}{(p+1)2^{p+1}}\varepsilon^{p+1}$.

Let $M \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$, where $M \cong \chi_S$ and $S = \{I_i\}_{i=1}^n$. For $1 \leq i \leq n$, let $x_i = (\inf I_i, \sup I_i)$. Choose $M' \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$ such that $\|\lambda(M) - \lambda(M')\|_p$ is sufficiently small. Let g be the minimizer in the definition of $W_{p+1}(M, M')$. For $1 \leq i \leq n$, let $\varepsilon_i = \|x_i - g(x_i)\|_\infty$. Then $\|\lambda(M) - \lambda(M')\|_p^p \geq \sum_{i=1}^n \frac{p+3}{(p+1)2^{p+1}}\varepsilon_i^{p+1} \geq \frac{p+3}{(p+1)2^{p+1}}W_{p+1}(M, M')^{p+1}$. \square

Following the same argument as in the proof of Corollary 3.12, we have the following.

Corollary 3.14. *Let $M \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$. Let $r > 0$. Then there exists r' such that $B_{r'}(M; d_p) \subseteq B_r(M; W_{p+1})$. \square*

4. STABILITY OF PERSISTENCE LANDSCAPES

In this section, we combine Proposition 3.10 and the stability theorems of [2] and [3] to get a stability theorem for persistence landscapes.

Let X be a triangulable compact metric space. For a function $f : X \rightarrow \mathbb{R}$, let $H_*(X_f, \mathbb{F})$ denote the (\mathbb{R}, \leq) indexed diagram of vector spaces given by $H_*(X_f, \mathbb{F})(a) = H_*(f^{-1}((\infty, a]), \mathbb{F})$, for all $a \in \mathbb{R}$, and $H_*(X_f, \mathbb{F})(a \leq b)$ is the map induced on homology by the inclusion $f^{-1}((\infty, a]) \subseteq f^{-1}((-\infty, b])$. We say that f is *tame* if $H_*(X_f, \mathbb{F}) \in \mathbf{Vec}_{\mathbf{ft}}^{(\mathbb{R}, \leq)}$. We abuse notation by denoting $H_*(X_f, \mathbb{F})$ by just f .

We assume that for some $k \geq 1$, there exists a constant C_X such that for all tame Lipschitz functions $f : X \rightarrow \mathbb{R}$ with Lipschitz constant, $\text{Lip}(f)$, at most 1, $\text{pers}_k(f) \leq C_X$ (see (8)).

Theorem 4.1 (Landscape stability theorem). *Let $f, g : X \rightarrow \mathbb{R}$ be two tame Lipschitz functions. Then*

$$d_p(f, g)^p \leq \text{pers}_1(f)\|f - g\|_\infty^p + \frac{2}{p+1}C\|f - g\|_\infty^{p+1-k},$$

where $C = C_X \max(\text{Lip}(f)^k, \text{Lip}(g)^k)$.

Proof. The theorem follows by applying the Main Theorem of [2] and the Wasserstein Stability Theorem of [3] to Proposition 3.10. \square

5. STATISTICAL INFERENCE WITH PERSISTENCE LANDSCAPES

In this section we prove a number of results useful for conducting statistical inference for persistent homology. We show that the spaces PL^p are complete and separable. For $p = 2$, we prove that the Fréchet mean and variance can be calculated pointwise and we give corresponding limit theorems.

5.1. Polish space. A good setting for probability theory is a metric space that is complete and separable. A metric allows one to consider convergence. Completeness allows one to use Cauchy sequences, and separability avoids certain measure-theoretic difficulties.

Recall that $\text{PL}^p = \text{PL} \cap \mathcal{L}^p(\mathbb{N} \times \mathbb{R}) \cong \text{PL} \cap L^p(\mathbb{N} \times \mathbb{R})$.

Proposition 5.1. *Let $1 \leq p < \infty$. Then PL^p is separable.*

Proof. For $1 \leq p < \infty$, $L^p(\mathbb{N} \times \mathbb{R})$ is a separable metric space. Since a subspace of a separable metric space is separable, the result follows. \square

Proposition 5.2. *Let $1 \leq p \leq \infty$. Then PL^p is complete.*

Proof. Let $\{f_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in PL^p . Since $\{f_i\}$ is a Cauchy sequence in $L^p(\mathbb{N} \times \mathbb{R})$, it is well known fact that $\{f_i\}$ converges pointwise a.e. and in $L^p(\mathbb{N} \times \mathbb{R})$ to a function $f \in L^p(\mathbb{N} \times \mathbb{R})$. In our case, each f_i is continuous, so in fact $\{f_i\}$ converges to f pointwise. Since f is the pointwise limit of 1-Lipschitz functions, it is 1-Lipschitz. It follows that $f \in \text{PL}^p$. \square

Combining the two propositions above, we have the following.

Theorem 5.3. *For $1 \leq p < \infty$, PL^p is a complete and separable metric space, i.e., it is a Polish space.* \square

5.2. Preliminaries. Here we prove some results that we will be useful later on. In this section, we assume that $1 \leq p < \infty$.

Lemma 5.4. *Let f and g be 1-Lipschitz functions. Let $t \in \mathbb{R}$. Then $\|f - g\|_p^p \geq \frac{1}{p+1} |f(t) - g(t)|^{p+1}$.*

Proof. Let $\alpha = |f(t) - g(t)|$. Since $|f - g|$ is 2-Lipschitz,

$$\begin{aligned} \|f - g\|_p^p &= \int |f(s) - g(s)|^p ds \geq \int_{t-\frac{\alpha}{2}}^{t+\frac{\alpha}{2}} |f(s) - g(s)|^p ds \\ &\geq 2 \int_0^{\frac{\alpha}{2}} (2u)^p du = \frac{\alpha^{p+1}}{p+1}. \quad \square \end{aligned}$$

Corollary 5.5. *Let f and g be 1-Lipschitz functions. Let $t \in \mathbb{R}$. Then $|f(t) - g(t)| < 2\|f - g\|_p^{\frac{p}{p+1}}$.*

Proof. By Lemma 5.4, $|f(t) - g(t)| \leq (p+1)^{\frac{1}{p+1}} \|f - g\|_p^{\frac{p}{p+1}}$. It is an easy calculus exercises to check that for $x > 0$, $x^{\frac{1}{x}}$ has a maximum at $x = e$, and $e^{\frac{1}{e}} < 2$. \square

Let $(k, t) \in \mathbb{N} \times \mathbb{R}$. Define $\text{ev}_{(k,t)} : \text{PL} \rightarrow \mathbb{R}$ by $\text{ev}_{(k,t)}(f) = f(k, t)$.

Lemma 5.6. *Let $1 \leq p < \infty$ and let $(k, t) \in \mathbb{N} \times \mathbb{R}$. Then the map $\text{ev}_{(k,t)} : \text{PL}^p \rightarrow \mathbb{R}$ is continuous.*

Proof. Choose $\varepsilon > 0$. Set $\delta = \left(\frac{1}{p+1}\varepsilon^{p+1}\right)^{\frac{1}{p}}$. Let $f, g \in \text{PL}^p$ such that $\|f - g\|_p < \delta$. By Lemma 5.4,

$$\|f - g\|_p^p = \sum_{i=1}^{\infty} \|f_i - g_i\|_p^p \geq \|f_k - g_k\|_p^p \geq \frac{1}{p+1} |f(k, t) - g(k, t)|^{p+1}.$$

Therefore $|f(k, t) - g(k, t)| < ((p+1)\delta^p)^{\frac{1}{p+1}} = \varepsilon$. Thus $\text{ev}_{(k,t)}$ is continuous. \square

Define $E : \mathbb{N} \times \mathbb{R} \times \text{PL}^p \rightarrow \mathbb{R}$ by $E(k, t, f) = f(k, t)$.

Lemma 5.7. *The map E is continuous.*

Proof. Choose $\varepsilon > 0$. Let $\delta = \min\left(1, \left(\frac{\varepsilon}{3}\right)^{\frac{p}{p+1}}\right)$. Assume $(k, t, f), (k', t', f') \in \mathbb{N} \times \mathbb{R} \times \text{PL}^p$ such that $\max(|k - k'|, |t - t'|, \|f - f'\|_p) < \delta$. Since $\delta \leq 1$, this implies that $k = k'$. Using the fact that f_k is 1-Lipschitz and Corollary 5.5,

$$\begin{aligned} |f(k, t) - f'(k, t')| &\leq |f(k, t) - f(k, t')| + |f(k, t') - f'(k, t')| \\ &\leq |t - t'| + 2\|f - f'\|_p^{\frac{p}{p+1}} < \delta + 2\delta^{\frac{p}{p+1}} \leq 3\delta^{\frac{p}{p+1}} = \varepsilon. \end{aligned}$$

Therefore E is continuous. \square

Since E is continuous, E is measurable. This will allow us to apply Fubini's theorem.

5.3. Fréchet mean and variance. A general notion of mean that can be defined in any metric space is the *Fréchet mean*, which we now define.

Let \mathcal{P} be a probability measure on $(\text{PL}^p, \mathcal{B})$ where $\mathcal{B} = \mathcal{B}(\text{PL}^p)$ is the σ -algebra of Borel sets in PL^p .

Definition 5.8. Define the *Fréchet function*, $F_{\mathcal{P}} : \text{PL}^p \rightarrow \mathbb{R}$ by

$$F_{\mathcal{P}}(f) = \int_{\text{PL}^p} \|f - g\|_p^2 \mathcal{P}(dg).$$

Define the *Fréchet variance* of \mathcal{P} by

$$\text{Var}_{\mathcal{P}} = \inf_{f \in \text{PL}^p} F_{\mathcal{P}}(f).$$

Define the *Fréchet mean set* of \mathcal{P} , to be the set $\{f \in \text{PL}^p \mid F_{\mathcal{P}}(f) = \text{Var}_{\mathcal{P}}\}$. If the Fréchet mean set contains a single element, we refer to that element as the *Fréchet mean* of \mathcal{P} .

Example 5.9. If \mathcal{P} is a probability measure on \mathbb{R} and $X \sim \mathcal{P}$, then the Fréchet mean of \mathcal{P} is $E[X]$, and the Fréchet variance of \mathcal{P} is $\text{Var}[X]$.

Let $1 \leq p < \infty$. Let $(k, t) \in \mathbb{N} \times \mathbb{R}$. By Lemma 5.6, $\text{ev}_{(k,t)} : (\text{PL}^p, \mathcal{B}, \mathcal{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable. So $\mathcal{P}_{(k,t)} := (\text{ev}_{(k,t)})_* \mathcal{P} = \mathcal{P} \text{ev}_{(k,t)}^{-1}$ is a probability measure on \mathbb{R} .

Definition 5.10. Define $h : \mathbb{N} \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ by $h(k, t) = E[X_{(k,t)}]$, where $X_{(k,t)} \sim \mathcal{P}_{(k,t)}$. That is, $h(k, t) = \int_{\mathbb{R}} s \mathcal{P}_{(k,t)}(ds) = \int_{\text{PL}^p} f(k, t) \mathcal{P}(df)$.

Lemma 5.11. $h \in \text{PL}$.

Proof. We show that $h \in \text{PL}$ by checking the three conditions in Lemma 2.8. Let $(k, t) \in \mathbb{N} \times \mathbb{R}$. Notice that $h(k, t) = \int s \mathcal{P}_{(k,t)}(ds)$. For (1), $\mathcal{P}_{(k,t)}(-\infty, 0) = \mathcal{P}\{f \in \text{PL}^p \mid f(k, t) < 0\} = 0$. Thus $h(k, t) = \int_0^\infty s \mathcal{P}_{(k,t)}(ds) \geq 0$. For (2), let $k \leq \ell \in \mathbb{N}$. Then

$$h(k, t) = \int_{\text{PL}^p} f(k, t) \mathcal{P}(df) \geq \int_{\text{PL}^p} f(\ell, t) \mathcal{P}(df) = h(\ell, t).$$

For (3), $|h(k, t) - h(k, s)| = \int_{\text{PL}^p} |f(k, t) - f(k, s)| \mathcal{P}(df) \leq \int_{\text{PL}^p} |t - s| \mathcal{P}(df) = |t - s|$. \square

Lemma 5.12. If $\int_{\text{PL}^p} \|f\|_p^p d\mathcal{P} < \infty$, then $h \in \mathcal{L}^p(\mathbb{N} \times \mathbb{R})$.

Proof. Using Fubini's theorem, which we can apply thanks to Lemma 5.7,

$$\begin{aligned} \|h\|_p^p &= \int_{\mathbb{N} \times \mathbb{R}} h(k, t)^p d\mu = \int_{\mathbb{N} \times \mathbb{R}} \int_{\text{PL}^p} f(k, t)^p d\mathcal{P} d\mu \\ &= \int_{\text{PL}^p} \int_{\mathbb{N} \times \mathbb{R}} f(k, t)^p d\mu d\mathcal{P} = \int_{\text{PL}^p} \|f\|_p^p d\mathcal{P} < \infty. \quad \square \end{aligned}$$

Remark 5.13. Since h_k is 1-Lipschitz by Lemma 5.11, from the proof of Lemma 5.12, we see that if $\int_{\text{PL}^p} \|f\|_p^p d\mathcal{P} < \infty$, then $h(k, t) < \infty$ for all k, t .

Lemma 5.14. *Assume $p=2$. Then $F_{\mathcal{P}}(f) = \int_{\mathbb{N} \times \mathbb{R}} F_{\mathcal{P}_{(k,t)}}(f(k, t)) d\mu$.*

Proof. By Fubini's Theorem, which we can apply using Lemma 5.7,

$$\begin{aligned} F_{\mathcal{P}}(f) &= \int_{\text{PL}^2} \int_{\mathbb{N} \times \mathbb{R}} (f(k, t) - g(k, t))^2 d\mu \mathcal{P}(dg) \\ &= \int_{\mathbb{N} \times \mathbb{R}} \int_{\text{PL}^2} (f(k, t) - g(k, t))^2 \mathcal{P}(dg) d\mu \\ &= \int_{\mathbb{N} \times \mathbb{R}} \int_{\mathbb{R}} (f(k, t) - y)^2 \mathcal{P}_{(k,t)}(dy) d\mu \\ &= \int_{\mathbb{N} \times \mathbb{R}} F_{\mathcal{P}_{(k,t)}}(f(k, t)) d\mu. \quad \square \end{aligned}$$

Theorem 5.15. *Assume $p = 2$ and $\int_{\text{PL}^p} \|f\|_p^p d\mathcal{P} < \infty$. Then the Fréchet mean of \mathcal{P} is h and the Fréchet variance of \mathcal{P} is $\int_{\mathbb{N} \times \mathbb{R}} \text{Var}_{\mathcal{P}_{(k,t)}} d\mu$.*

Proof. By Example 5.9, $\mathcal{P}_{(k,t)}$ has Fréchet mean $h(k, t)$. Applying this to Lemma 5.14, we get that $F_{\mathcal{P}}$ has Fréchet mean h . Furthermore, again combining Lemma 5.9 and Lemma 5.14, $F_{\mathcal{P}}(h) = \int_{\mathbb{N} \times \mathbb{R}} \text{Var}_{\mathcal{P}_{(k,t)}} d\mu$. \square

5.4. Limit theorems. In this section we prove some pointwise limit theorems for persistence landscapes.

Let \mathcal{P} be a probability measure on the space $\text{PL}^2 = \text{PL} \cap \mathcal{L}^2(\mathbb{R}^2)$, with $\int_{\text{PL}^p} \|f\|_p^p d\mathcal{P} < \infty$. Then by Theorem 5.15, $h_{\mathcal{P}}$ is the Fréchet mean of \mathcal{P} .

Let $\lambda_1, \dots, \lambda_n \in \text{PL}^2$ be an iid sample from \mathcal{P} . Let $\hat{\mathcal{P}} = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i}$ denote the empirical measure. Then by Definition 5.10, for all $(k, t) \in \mathbb{N} \times \mathbb{R}$, $h(k, t) = \frac{1}{n} \sum_{i=1}^n \lambda_i(k, t)$. So by Theorem 5.15, $h_{\hat{\mathcal{P}}} = \frac{1}{n} \sum_{i=1}^n \lambda_i =: \bar{\lambda}_n$ is the Fréchet mean of $\hat{\mathcal{P}}$. By Theorem 5.15, the Fréchet variance of $\hat{\mathcal{P}}$ is given by $\text{Var}_{\hat{\mathcal{P}}} = \frac{1}{n} \|\bar{\lambda}_n - \lambda_i\|_2^2$.

Now fix $(k, t) \in \mathbb{N} \times \mathbb{R}$. For $1 \leq i \leq n$, let $X_i = \lambda_i(k, t)$. Let $\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i = \bar{\lambda}_n(k, t)$ and $\mu := h_{\mathcal{P}}(k, t)$. By Remark 5.13, $\mu < \infty$. Also, $E[X_i] = \mu$. Recall that $\text{ev}_{(k,t)}(\lambda) = \lambda(k, t)$ and $\mathcal{P}_{(k,t)} = \mathcal{P} \text{ev}_{(k,t)}^{-1}$. Thus X_1, \dots, X_n is an iid sample of integrable random variables from $\mathcal{P}_{(k,t)}$ with $E[X_i] = \mu < \infty$. So we can apply the Strong Law of Large Numbers.

Theorem 5.16 (Strong Law of Large Numbers). *As $n \rightarrow \infty$,*

$$\bar{X}_n \xrightarrow{a.s.} \mu.$$

That is, $\Pr(\lim_{n \rightarrow \infty} \bar{X}_n = \mu) = 1$. \square

Corollary 5.17 (Pointwise convergence of the sample mean to the Fréchet mean). *For all $(k, t) \in \mathbb{N} \times \mathbb{R}$,*

$$\bar{\lambda}_n(k, t) \xrightarrow{a.s.} h_{\mathcal{P}}(k, t).$$

\square

Now assume that X_i has finite variance. Let $\sigma^2 = \text{Var}[X_i]$. Then we have the following from the Central Limit Theorem. Let $N(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2 . Let Φ denote the standard normal cumulative distribution function, $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$.

Theorem 5.18 (Central Limit Theorem). *As $n \rightarrow \infty$,*

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2).$$

That is, $\forall r \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \Pr(\sqrt{n}(\bar{X}_n - \mu) \leq r) = \Phi\left(\frac{r}{\sigma}\right)$. \square

Corollary 5.19 (Pointwise CLT for the sample mean). *For all $(k, t) \in \mathbb{N} \times \mathbb{R}$,*

$$\sqrt{n}(\bar{\lambda}_n(k, t) - h_{\mathcal{P}}(k, t)) \xrightarrow{d} N(0, \sigma^2).$$

\square

Recall that the Fréchet variance of the empirical measure is given by $\text{Var}_{\hat{\mathcal{P}}} = \frac{1}{n} \sum_{i=1}^n \|\lambda_i - \bar{\lambda}_n\|_2^2 = \frac{1}{n} \int (\lambda_i(k, t) - \bar{\lambda}_n(k, t))^2 d\mu$. So,

$$\text{Var}_{\hat{\mathcal{P}}} = \int s_n^2(k, t) d\mu,$$

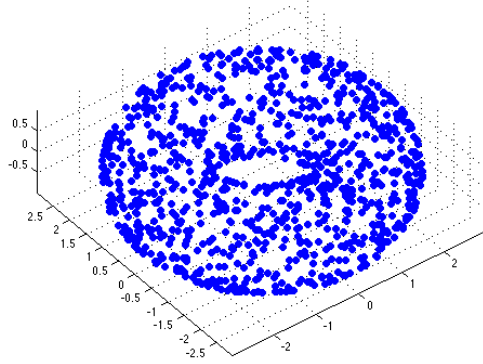
where $s_n^2(k, t) = \frac{1}{n} \sum_{i=1}^n (\lambda_i(k, t) - \bar{\lambda}_n(k, t))^2$ is the biased sample variance. Using $s^2(k, t) = \frac{1}{n-1} \sum_{i=1}^n (\lambda_i(k, t) - \bar{\lambda}_n(k, t))^2$, the unbiased sample variance, we have

$$\text{Var}_{\hat{\mathcal{P}}} = \frac{n-1}{n} \int s^2(k, t) d\mu.$$

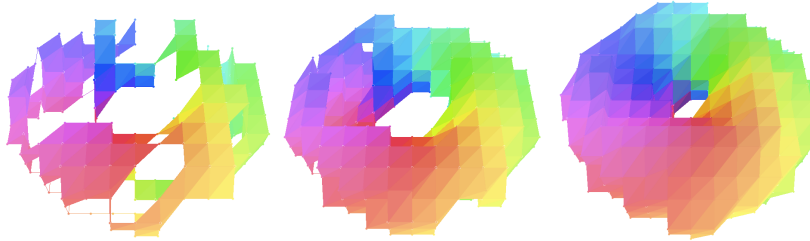
5.5. Interpretation of the average persistence landscape. For a sample $\lambda_1, \dots, \lambda_n$ of persistence landscapes, we call the empirical Fréchet mean, $\bar{\lambda}_n$, the *average persistence landscape*. Consider barcodes B_1, \dots, B_n and the corresponding persistence landscapes $\lambda_1, \dots, \lambda_n$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$, $\bar{\lambda}_n(k, t)$ is the average value of the largest radius interval centered at t that is contained in k intervals in the barcodes B_1, \dots, B_n .

6. EXAMPLES

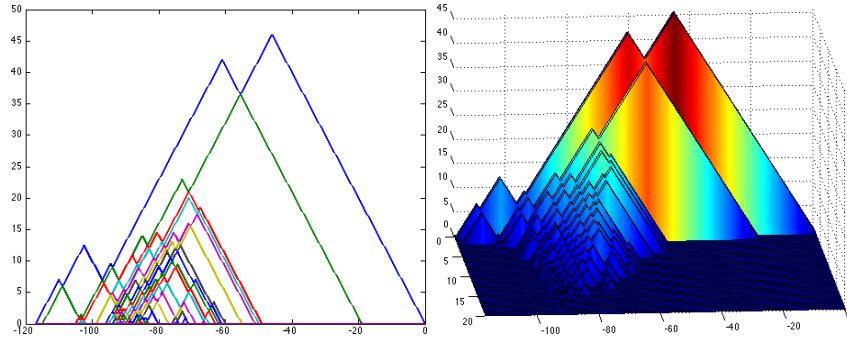
6.1. Persistence landscape of a torus. Consider an iid sample of 1000 points from a torus embedded in \mathbb{R}^3 using the uniform surface area measure [4].



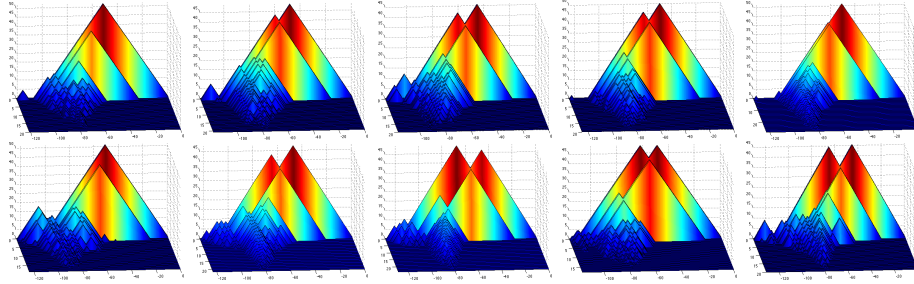
For these points, we construct a filtered simplicial complex as follows. First we triangulate the underlying space using the Coxeter–Freudenthal–Kuhn triangulation. Next we smooth our data using a triangular kernel. We evaluate this kernel density estimator at the vertices of our simplicial complex. Finally, we filter our simplicial complex by taking the flag complex on lower excursion sets of the vertices. That is, for filtration level r , we include a simplex in our triangulation if and only if the kernel density estimator has values less than or equal to r at all of its vertices. Here are three stages in the filtration.



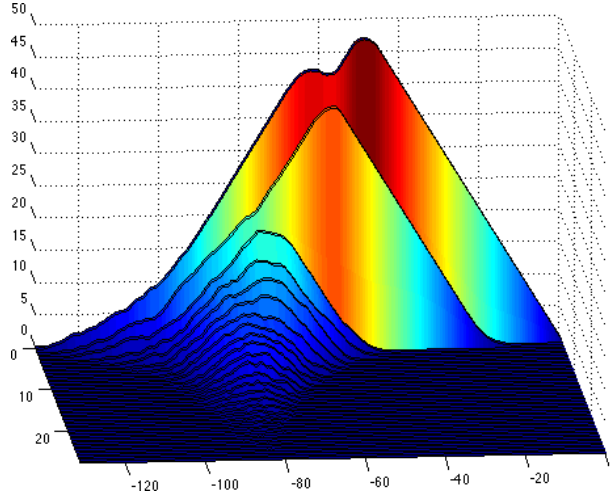
We then calculate the persistent homology of this filtered simplicial complex using, say, javaPlex [8]. For the persistent homology in dimension 1, here is the corresponding persistence landscape, in both 2d and 3d versions.



6.2. Average persistence landscape of a torus. Now if we repeat this 10 times, we have a sample X_1, X_2, \dots, X_{10} of ten persistence landscapes.

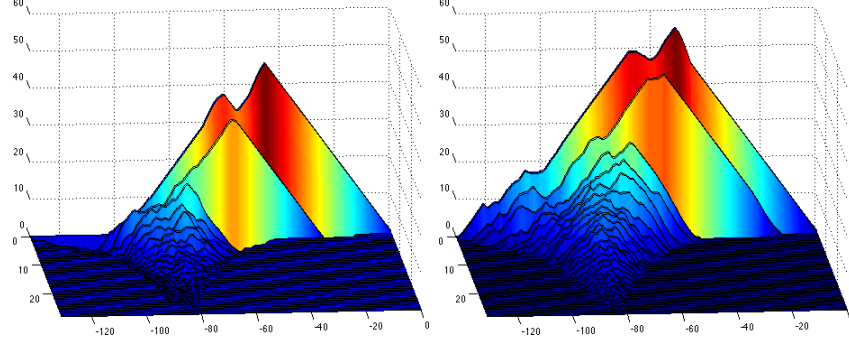


We calculate their Fréchet mean by averaging pointwise.

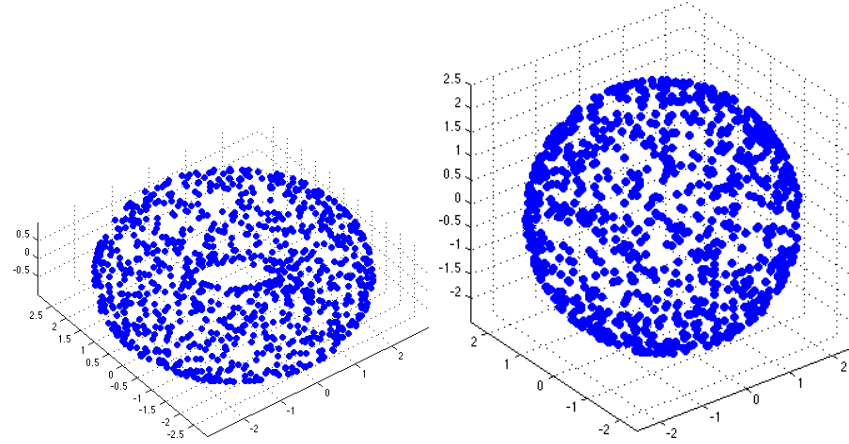


We remark that this average persistence landscape is not the persistence landscape corresponding to a barcode or persistence diagram. However, there is a direct interpretation of the average persistence landscape in terms of barcodes (see Section 5.5).

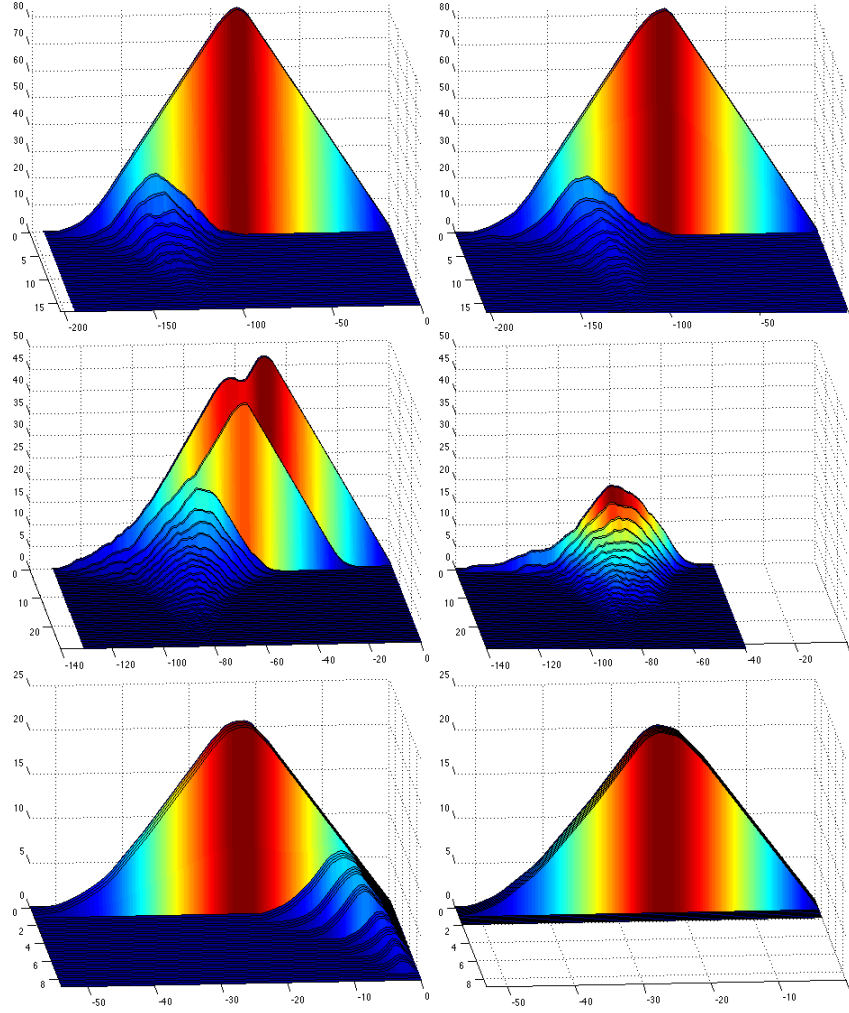
We can get a sense of the stability of the features of this average persistence landscape by subtracting (on the left) and adding (on the right) 2 standard deviations pointwise.



6.3. Comparing landscapes for a torus and a sphere. Now let us apply statistical inference to persistence landscapes obtained from a torus and sphere with the same surface area.



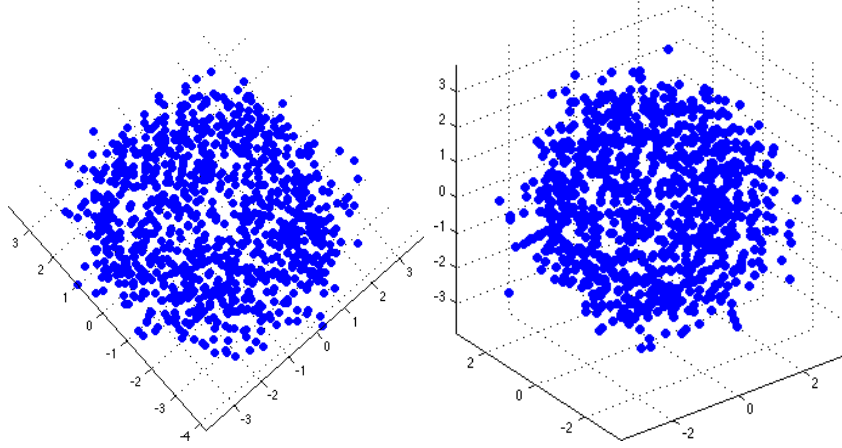
For $k = 0, 1, 2$, let $X_1^k, X_2^k, \dots, X_{10}^k$ be an iid sample of persistence landscapes in dimension k obtained from 1000 iid points on a torus sampled using the uniform surface area measure. Let $Y_1^k, Y_2^k, \dots, Y_{10}^k$ be the corresponding sample of persistence landscapes from a sphere. Here are the average persistence landscapes. In rows 1, 2 and 3, we have the average persistence landscape in dimension 0, 1 and 2 of the torus on the left and the sphere on the right.



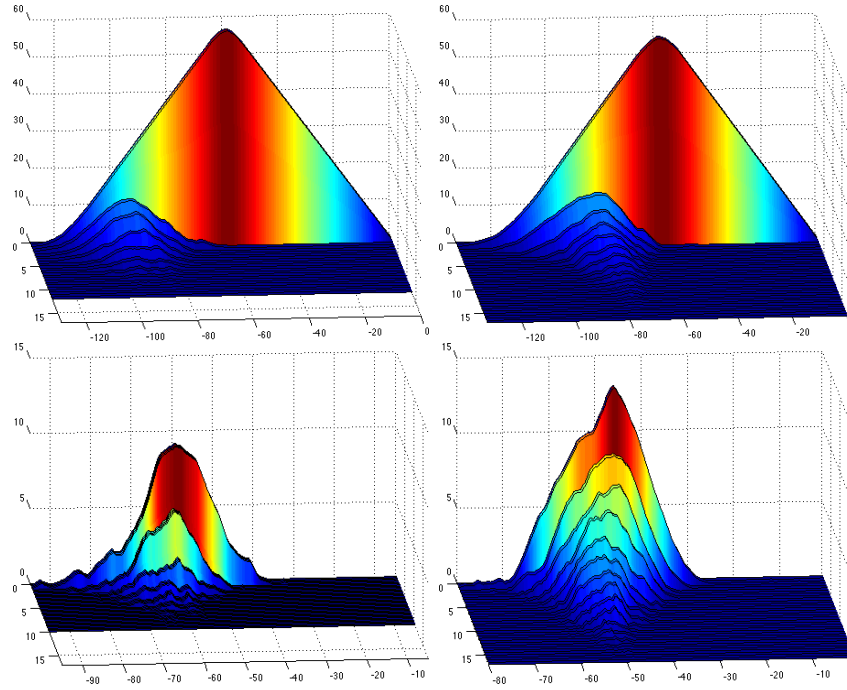
In dimension 0, the d_2 distance between the average landscapes of the torus and sphere is 48.8 and the square root of the Fréchet variances of the torus and sphere are 123.4 and 143.8, respectively. In dimension 1, we obtain a distance of 370.0 and square root of Fréchet variances of 69.6 and 57.6. In dimension 2, these numbers are 26.6, 23.7, and 24.8.

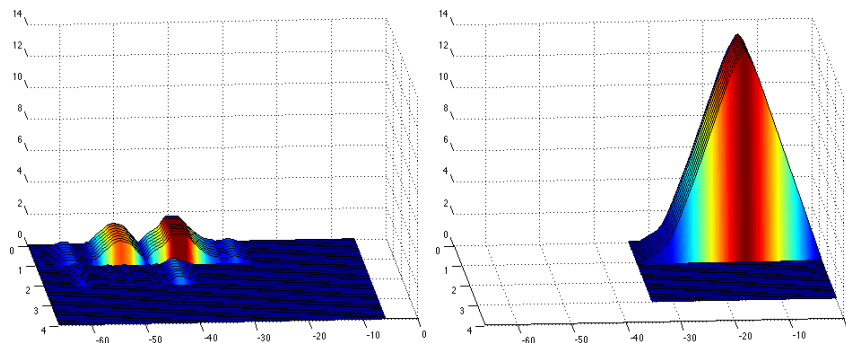
We use the permutation test with 10,000 repetitions to determine if the difference between the two samples is statistically significant. Comparing the two samples for dimension 0, the difference is not significant (p value 0.6106). In dimension 1, the difference is significant with a p value of 0.0000. Restricting to each of $k = 1$ and $k = 2$ we have p values of 0.0000 and 0.0217, respectively. Restricting to $k > 2$, there is no significant difference (p value 0.6570). In dimension 2, we get a p value of 0.0088. However if we restrict to $k = 1$ the difference is not significant (p value 0.9997).

6.4. Comparing landscapes from noisy samples. We now repeat the analysis of the previous section with the addition of Gaussian noise to the point samples. On the left we have 1000 points sampled from a torus, from the perspective that makes it easiest to see the hole in the middle. On the right we have points sampled from the sphere.



We calculate persistent homology in the same way as in Section 6.3. Here are the average persistence landscapes. In rows 1, 2 and 3, we have the average persistence landscape in dimension 0, 1 and 2, respectively, with the torus on the left and the sphere on the right.

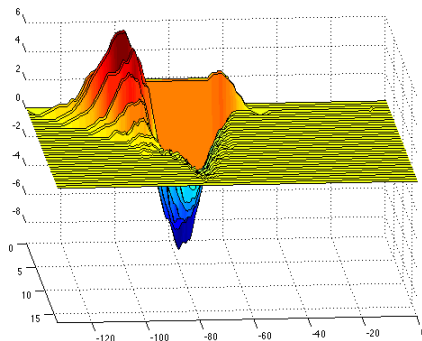




Again we calculate the d_2 distances between the average landscapes and the square root of the Fréchet variances from the torus and the sphere. In dimension 0, they are 69.7, 67.3 and 78.1. In dimension 1, we have 59.5, 23.5 and 26.4. In dimension 2, the values are 44.5, 7.2 and 10.2.

Again, we use the permutation test to determine if the difference between the two samples is statistically significant.

There is a significant difference in dimension 0, with a p value of 0.0111. This is surprising, since the average landscapes look very similar. However, on closer inspection, they are shifted slightly. Here is a graph of the difference.



Note that we are detecting a geometric difference, not a topological one. Less surprisingly, there is also a significant difference in dimensions 1 and 2, with p values of 0.0000 and 0.0000, respectively.

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